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Even the First Iterate of a Markov Operator Is Contracting in an L_2 Norm

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A weighted L_2 norm is introduced in which Markov operators, e.g., associated with noisy maps, are contracting provided the kernel (i.e., the transitional distribution) is smooth enough. This results in strong relaxational properties of noisy maps. Similar to this norm, integral functionals appear useful when studying spatiotemporal chaos and random fields.

KEY WORDS: Markov operator; invariant distribution; relaxation; noisy map; random field.

1. INTRODUCTION

Recent years have brought remarkable successes in understanding chaotic attractors. However, not long ago it appeared that sometimes it is not the equilibrium, but the *transient* regime which is of importance.[1,2] Moreover, apart from its own significance for describing transients (as to how the dynamical system, once started, converges to the stationary regime), the understanding of relaxation is very important for the perturbation theory, even that of *attractors*. Indeed, the zeroth approximation usually treats a complex system as a group of noninteracting subsystems. The higher ones "turn on" the (weak) interactions, so the subsystems are not exactly on their attractors. The deviation from the zeroth approximation appears as a balance between the interactions, pulling the subsystems from their attractors, and their relaxations toward them.[3-5]

In this paper we will investigate relaxation toward the statistical equilibrium in the noisy map

$$x_{n+1} = f(x_n) + \zeta_n \tag{1.1}$$

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where ζ is a random noise with distribution $w(\zeta)$. This system originates a Markov chain, and the distribution at the nth iteration $p_n(x)$ obeys[6,7]

$$p_{n+1}(x) = (\mathscr{L}_{f,w}\, p_n)(x) \equiv \int w(x - f(y))\, p_n(y)\, dy \tag{1.2}$$

If the external noise vanishes, i.e., its distribution becomes a δ-function, the master equation (1.2) takes the form

$$p_{n+1}(x) = (\mathscr{L}_f\, p_n)(x) \equiv \int \delta(x - f(y))\, p_n(y)\, dy \tag{1.3}$$

and is called the Frobenius–Perron equation.[8]

The relaxation toward the statistical equilibrium means a convergence to the *invariant* distribution $P(x)$: $p_n \xrightarrow{n \to \infty} P$. This convergence was extensively studied, and the well-known ergodic theorems were proved which state that (under not too strong limitations) the iterates of two different initial distributions $\tilde{p}_0$ and p_0 converge asymptotically exponentially:

$$\|\Delta p_n\|_{L_1} \leqslant \begin{cases} \|\Delta p_0\|_{L_1} & \text{for any} \quad n \\ \kappa^n \cdot \|\Delta p_0\|_{L_1} & \text{for} \quad n \geqslant n_0 \end{cases}$$

where $\Delta p_n \equiv \tilde{p}_n - p_n$, $\kappa = \text{const} < 1$.[7] In other words, the iterates of the master operator $\mathscr{L}_{f,w}$ are *asymptotically* contracting:

$$\|\mathscr{L}^n_{f,w}\|_{L_1} \leqslant \begin{cases} 1 & \text{for any} \quad n \\ \kappa^n & \text{for} \quad n \geqslant n_0 \end{cases} \tag{1.4}$$

A relation similar to (1.4) was also proved for a Frobenius–Perron operator $\mathscr{L}_f$ (i.e., for the noiseless case),[9,10] though this required strong limitations on f (e.g., in the one-dimensional case it should be piecewise differentiable with $|f'| > 2$, etc.). Unfortunately, now n_0 *depends* on Δp_0, and goes to infinity when Δp_0 becomes singular.[3] I mention also an interesting investigation of the spectral characteristics of $\mathscr{L}_f$ for some simple maps f done in ref. 11.

The convergence (1.4) requires that $\Delta p(x)$ be, in a sense, "equidistributed" over parts of supp P; otherwise the sequence Δp_n will be asymptotically periodic (see Ref. 7). Say, let supp P be a union of two intervals I_1 and I_2; and denote $P^{(k)}(x) \equiv P(x) \cdot \chi_{I_k}(x)$, where $\chi_A(x)$ is the indicator of the set A, i.e., 1 for $x \in A$ and 0 otherwise. Obviously, the Markov chain permutes these $P^{(k)}$: $\mathscr{L}_{f,w} P^{(1)} = P^{(2)}$, $\mathscr{L}_{f,w} P^{(2)} = P^{(1)}$, thus for $\Delta p_0 = P^{(1)} - P^{(2)}$ (notice that $\int \Delta p_0\, dx = 0$) we have $\mathscr{L}_{f,w}\, \Delta p_0 = P^{(2)} - P^{(1)} = -\Delta p_0$, etc., and the sequence $\Delta p_n(x) = (-1)^n\, \Delta p_0(x)$ is not converging but

2-periodic. Considering the second iterate $\mathscr{L}^2_{f,w}$ one easily finds that convergence is recovered if, and only if, $\int_{I_k} \Delta p_0(x)\, dx = 0$. The same holds when the attractor consists of a greater number of intervals.

Frequently the asymptotic estimate (1.4) is sufficient,[2-5] but sometimes it is crucial that already the first iterate of the master operator itself be contracting. In the L_1 norm this is impossible; one easily finds that for Δp_0 consisting of *narrow* peaks only $\|\mathscr{L}_{f,w} \Delta p_0\|_{L_1} = \|\Delta p_0\|_{L_1}$. In this paper we will prove that in the weighted L_2 norm

$$\|\Delta p\|^2 \equiv \int \frac{[\Delta p(x)]^2}{P(x)}\, dx \tag{1.5}$$

where $P(x)$ is the invariant distribution, even the first iterate of the Markov operator (1.2) is contracting: $\|\mathscr{L}_{f,w}\| \leqslant \kappa < 1$ [for those Δp for which the integral (1.5) exists and $\int_{I_k} \Delta p\, dx = 0$, where I_k is any interval from those composing $\operatorname{supp} P = \bigcup_{i=1}^{N} I_i$]. Our proof requires that the map $x_{n+1} = f(x_n)$ possess a bounded attractor, and the distribution of the noise $w(\cdot)$ has (bounded) derivatives up to the fourth. The latter condition seems superfluous for an *integral* operator and apparently results from this way of proof.

In conclusion we discuss the significance of L_2 norms closely related with (1.5) for spatiotemporal chaos and random fields.

2. THE GENERAL ESTIMATES

Let us show that the Markov operator with a bounded kernel is not expanding in the norm (1.5). The Markov operator is a linear integral operator

$$(Kp)(x) \equiv \int k(x, y)\, p(y)\, dy \tag{2.1}$$

whose kernel $k(\cdot, \cdot)$ is generally the transitional probability and so satisfies $k \geqslant 0$, $\int k(x, y)\, dy = 1$.

Here and below the (2.1)*-type operator will be denoted by the same letter as its kernel, only in upper case.*

Now take some $p(x) \geqslant 0$ and an arbitrary $\Delta p(x)$ for which $\int \{[\Delta p(x)]^2/p(x)\}\, dx$ exists. This obviously implies that $\operatorname{supp} \Delta p \subseteq \operatorname{supp} p$; outside, where this fraction is undefined, we put $[\Delta p]^2/p \equiv 0$, so here and below

$$\operatorname{supp} \frac{[\Delta p]^2}{p} \subseteq \operatorname{supp} p \tag{2.2}$$

Then

$$\int \frac{[\Delta p(y)]^2}{p(y)} k(x, y)\, dy$$

exists and by the Cauchy–Bunjakowsky inequality

$$\begin{aligned}((K\Delta p)(x))^2 &= \left(\int \frac{\Delta p(y)}{p(y)} k(x, y)\, p(y)\, dy\right)^2 \\ &\leqslant \int k(x, y)\, p(y)\, dy \cdot \int \left(\frac{\Delta p(y)}{p(y)}\right)^2 k(x, y)\, p(y)\, dy \\ &= (Kp)(x) \int \frac{[\Delta p(y)]^2}{p(y)} k(x, y)\, dy\end{aligned}$$

so

$$\frac{((K\Delta p)(x))^2}{(Kp)(x)} \leqslant \int \frac{[\Delta p(y)]^2}{p(y)} k(x, y)\, dy \equiv \left(K \circ \frac{[\Delta p]^2}{p}\right)(x) \tag{2.3}$$

which after integration gives

$$\int \frac{((K\Delta p)(x))^2}{(Kp)(x)}\, dx \leqslant \int \frac{[\Delta p(y)]^2}{p(y)}\, dy \tag{2.4}$$

Notice that, comparing (2.3) with (2.1) and recalling that $\operatorname{supp}([\Delta p]^2/p) \subseteq \operatorname{supp} p$, we easily conclude that

$$\operatorname{supp} \frac{[K\Delta p]^2}{Kp} \subseteq \operatorname{supp} Kp \tag{2.5}$$

All this obviously holds for the operator $\mathscr{L}_{f,w}$ of (1.2), so taking for $p(x)$ its invariant distribution $P = \mathscr{L}_{f,w} P$ and using the norm (1.5), we get

$$\|\mathscr{L}_{f,w}\, \Delta p\| \leqslant \|\Delta p\|$$

or

$$\|\mathscr{L}_{f,w}\| \leqslant 1$$

Notice that these estimates are also valid for the Frobenius–Perron operator (1.3), though its kernel $\delta(x - f(y))$ is singular. Indeed, the existence of $\int \{[\Delta p(x)]^2/p(x)\}\, dx$ implies that $[\Delta p(x)]^2/p(x)$ is bounded for almost all x, thus

$$\int \frac{[\Delta p(y)]^2}{p(y)} \delta(x - f(y))\, dy$$

which is in fact a *sum* over the preimages $f^{-1}(x)$, exists for almost all x. Therefore,

$$\int \frac{((\mathscr{L}_f \Delta p)(x))^2}{(\mathscr{L}_f p)(x)} dx \leqslant \int \frac{[\Delta p(y)]^2}{p(y)} dy \tag{2.6}$$

Unfortunately, $\mathscr{L}_f$ is not contracting even in the norm (1.5), and one can easily construct p and Δp for which (2.6) is an equality.

Comparing (1.2) with (1.3), one finds that $\mathscr{L}_{f,w}$ is a composition: $\mathscr{L}_{f,w} = W\mathscr{L}_f$, where

$$(Wp)(x) \equiv \int w(x-y)\, p(y)\, dy \tag{2.7}$$

and since $\mathscr{L}_f$ satisfies (2.4), it suffices to prove that

$$\int \frac{((W\Delta \rho)(x))^2}{(W\rho)(x)} dx < \int \frac{[\Delta \rho(x)]^2}{\rho(x)} dx$$

to show that $\mathscr{L}_{f,w}$ is contracting in the norm (1.5). This is the main content of this paper. For the sake of simplicity we will consider the one-dimensional case; the multidimensional generalization is straightforward and consists mainly of technicalities, such as matrices of derivatives, etc.

3. THE IDEA OF THE PROOF

The idea of the proof comes from the properties of the (2.7)-like convolution operator Q_ε whose kernel $q_\varepsilon(\cdot)$ is a narrow peak. Throughout the paper we will use

$$q_\varepsilon(\xi) \equiv \begin{cases} 1/2\varepsilon & \text{if } |\xi| \leqslant \varepsilon \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

Let us introduce the relative deviation $u \equiv \Delta p/p$ and assume that in some domain D' it is smooth. Then, we write

$$(Q_\varepsilon \Delta p)(x) \equiv \int \Delta p(y)\, q_\varepsilon(x-y)\, dy = \int u(x+\xi)\, q_\varepsilon(\xi)\, p(x+\xi)\, d\xi \tag{3.2}$$

where, owing to (3.1), the integration goes over $|\xi| \leqslant \varepsilon$. When $x \in D'$ we expand $u(x+\xi)$ and $p(x+\xi)$ in the Taylor series

$$u(x+\xi) = u(x) + \xi u'(x) + \xi^2 u''(x)/2 + O(\varepsilon^3)$$

$$p(x+\xi) = p(x) + \xi p'(x) + O(\varepsilon^2)$$

and substituting these expansions in (3.2), we easily get

$$(Q_\varepsilon \Delta p)(x) = u(x)(Q_\varepsilon p)(x) + \frac{\varepsilon^2}{6}(u''(x)\, p(x) + 2u'(x)\, p'(x)) + O(\varepsilon^3), \quad x \in D' \tag{3.3}$$

Similarly,

$$\begin{aligned} \int \frac{[\Delta p(y)]^2}{p(y)} q_\varepsilon(x-y)\, dy &= \int u^2(x+\xi)\, q_\varepsilon(\xi)\, p(x+\xi)\, d\xi \\ &= u^2(x)(Q_\varepsilon p)(x) + \frac{\varepsilon^2}{6}((u^2)''(x)\, p(x) + 2(u^2)'(x)\, p'(x)) \\ &\quad + O(\varepsilon^3), \qquad x \in D' \end{aligned}$$

which together with (3.3) yields that for $x \in D'$

$$\frac{((Q_\varepsilon \Delta p)(x))^2}{(Q_\varepsilon p)(x)} = \int \frac{[\Delta p(y)]^2}{p(y)} q_\varepsilon(x-y)\, dy - \frac{\varepsilon^2}{3} p(x)(u'(x))^2 + O(\varepsilon^3) \tag{3.4}$$

Outside D', where the Taylor expansions fail, the estimate (2.3) still gives

$$\frac{((Q_\varepsilon \Delta p)(x))^2}{(Q_\varepsilon p)(x)} \leqslant \int \frac{[\Delta p(y)]^2}{p(y)} q_\varepsilon(x-y)\, dy$$

Combining it with (3.4) and integrating, we have

$$\int \frac{((Q_\varepsilon \Delta p)(x))^2}{(Q_\varepsilon p)(x)}\, dx \leqslant \int \frac{[\Delta p(y)]^2}{p(y)}\, dy - \frac{\varepsilon^2}{3} \int_{D'} p(x)(u'(x))^2\, dx + O(\varepsilon^3) \tag{3.5}$$

so for ε small enough the operator Q_ε is indeed contracting.

The forthcoming program is the following. First we build a bridge between the operator W of (2.7) and Q_ε by proving that if the kernel w is smooth, then for any ε small enough

$$W = Q_\varepsilon H_\varepsilon + O(\varepsilon^3) \tag{3.6}$$

where H_ε is the (2.7)-type convolution operator with bounded nonnegative kernel h_ε. According to (2.4), it is not expanding, while Q_ε is, as suggested by (3.5), contracting. Hence for the product $W_\varepsilon = Q_\varepsilon H_\varepsilon$ we have

$$\int \frac{((W_\varepsilon \Delta p)(x))^2}{(W_\varepsilon p)(x)}\, dx \leqslant \int \frac{[\Delta p(y)]^2}{p(y)}\, dy - O(\varepsilon^2) + O(\varepsilon^3) \tag{3.7}$$

so taking ε small enough and recalling that $W = W_\varepsilon + O(\varepsilon^3)$, we conclude that W is contracting.

Sections 4–5 contain proofs of these outlines.

4. THE DECOMPOSITION (3.6)

In terms of the kernels, (3.6) means that there exists $0 \leqslant h_\varepsilon(x) < \infty$ with $\int h_\varepsilon \, dx = 1$ and such that

$$w_\varepsilon(x) \equiv \int q_\varepsilon(x-y)\, h_\varepsilon(y)\, dy = \int q_\varepsilon(\xi)\, h_\varepsilon(x+\xi)\, d\xi \tag{4.1}$$

which is the kernel of the product operator W_ε, is close to $w(x)$:

$$|w - w_\varepsilon| \leqslant O(\varepsilon^3)$$

If h_ε admits the Taylor expansion

$$h_\varepsilon(x+\xi) = h_\varepsilon(x) + \xi h'_\varepsilon(x) + \xi^2 h''_\varepsilon(x)/2 + O(\varepsilon^3)$$

then substituting it in (4.1), one easily calculates that $w_\varepsilon = h_\varepsilon + \frac{1}{6}\varepsilon^2 h''_\varepsilon + O(\varepsilon^3)$, so to obtain $w_\varepsilon = w + O(\varepsilon^3)$ it is natural to try $h_\varepsilon = h_\varepsilon^{(0)} \equiv w - \varepsilon^2 w''/6$. Below we will check whether it is indeed the sought-for solution.

Let us assume that $w(x)$ has continuous derivatives up to the fourth:

$$\left| \frac{d^n}{dx^n} w(x) \right| \leqslant \mathscr{W}_n \leqslant \mathscr{W}/2, \qquad n = 0,..., 4 \tag{4.2}$$

and that supp w is a single interval $[x_0, x_1]$ on which w does not vanish (at the end of this section we will show how to eliminate this limitation).

Consider $w_\varepsilon^{(0)}(x)$:

$$w_\varepsilon^{(0)}(x) \equiv \int q_\varepsilon(x-y)\, h_\varepsilon^{(0)}(y)\, dy$$

$$= \int q_\varepsilon(\xi)\, w(x+\xi)\, d\xi - \frac{\varepsilon^2}{6} \int q_\varepsilon(\xi)\, w''(x+\xi)\, d\xi$$

Expanding $w(x+\xi)$ and $w''(x+\xi)$ in the *exact* Taylor series with the Lagrange remainders

$$w(x+\xi) = w(x) + \xi w'(x) + \xi^2 w''(x)/2 + \xi^3 w'''(x + \xi^*[x, \xi])/6$$

$$w''(x+\xi) = w''(x) + \xi w'''(x + \xi^{**}[x, \xi])$$

where ξ^* and ξ^{**} are intermediate points between 0 and ξ, we get

$$w_\varepsilon^{(0)}(x) = w(x) + \frac{1}{6}\int \xi^3 q_\varepsilon(\xi)\, w'''(x+\xi^*)\, d\xi - \frac{\varepsilon^2}{6}\int \xi q_\varepsilon(\xi)\, w'''(x+\xi^{**})\, d\xi$$

and according to (4.2),

$$|w_\varepsilon^{(0)}(x) - w(x)| \leqslant \frac{\mathscr{W}}{12}\int |\xi|^3\, q_\varepsilon(\xi)\, d\xi + \frac{\mathscr{W}}{12}\varepsilon^2 \int |\xi|\, q_\varepsilon(\xi)\, d\xi = \frac{\varepsilon^3 \mathscr{W}}{16} \tag{4.3}$$

Then, since $w(x)$ is smooth and its support bounded, $\int w''\, dx$ vanishes and so $\int h_\varepsilon^{(0)}\, dx = \int w\, dx$. As w is the distribution of noise, $\int w\, dx = 1$ and thus $\int h_\varepsilon^{(0)}\, dx = 1$. Therefore, $h_\varepsilon^{(0)}$ would be just what we need unless this $h_\varepsilon^{(0)}(x)$ may *turn negative* [near the boundaries of supp w, where $w \leqslant O(\varepsilon^2)$], thus making $H_\varepsilon^{(0)}$ not Markovian. This may occur *only* for those x where

$$w(x) \leqslant \varepsilon^2 \mathscr{W}_2/6 \leqslant \varepsilon^2 \mathscr{W}/12 \tag{4.4}$$

In fact, the "dangerous" interval is even narrower. Indeed, consider the behavior of $h_\varepsilon^{(0)}(x)$ near the "dangerous" endpoints of supp w, i.e., x_0 and x_1. Assume that near these ends x_i, $w(x)$ behaves like a power-law function,

$$\begin{aligned} w(x) &= a_i \cdot (x-x_i)^{k_i} + o((x-x_i)^{k_i}) \\ w''(x) &= a_i \cdot k_i(k_i-1)(x-x_i)^{k_i-2} + o((x-x_i)^{k_i-2}) \end{aligned} \tag{4.5}$$

where, according to the smoothness condition (4.2), $k_i \geqslant 4$. Since according to (4.4) the "dangerous" domain *shrinks* for $\varepsilon \to 0$, we may use the asymptotic (4.5), which gives

$$\begin{aligned} h_\varepsilon^{(0)}(x) = {} & a_i(x-x_i)^{k_i-2}\, [(x-x_i)^2 - k_i(k_i-1)\,\varepsilon^2] \\ & + o((x-x_i)^{k_i-2}\, [(x-x_i)^2 + \varepsilon^2]) \end{aligned} \tag{4.6}$$

from which it follows that in fact $h_\varepsilon^{(0)}$ turns negative for (and *only* for) those x where

$$|x - x_i| \leqslant \varepsilon[k_i(k_i-1)]^{1/2} + o(\varepsilon)$$

and for h_ε to be nonnegative it has to differ from $h_\varepsilon^{(0)}$ in this narrow domain. Since according to (4.6) in this domain $|h_\varepsilon^{(0)}| \leqslant O(\varepsilon^k)$, where $k \equiv \min k_i$, the necessary modification is only slight, $O(\varepsilon^k)$. This enables us to define h_ε as follows. First we introduce $h_\varepsilon^{(1)}$ as (see Fig. 1):

1. In $[x_0 + 2\varepsilon\{k_0(k_0-1)\}^{1/2}, x_1 - 2\varepsilon\{k_1(k_1-1)\}^{1/2}]$

$$h_\varepsilon^{(1)}(x) = h_\varepsilon^{(0)}(x) \equiv w(x) - \varepsilon^2 w''(x)/6$$

2. Outside $[x_0 + \varepsilon, x_1 - \varepsilon]$, $h_\varepsilon^{(1)}(x) \equiv 0$.

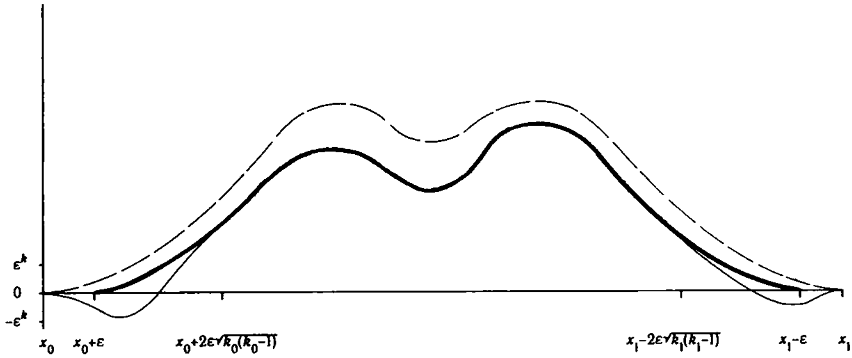

Fig. 1. A qualitative sketch of $w(x)$ (dashed line), $h_\varepsilon^{(0)}(x)$ (solid line), and $h_\varepsilon^{(1)}(x)$ (bold line).

3. In the remaining domain, i.e., for $[x_0+\varepsilon, x_0+2\varepsilon\{k_0(k_0-1)\}^{1/2}]$ and $[x_1-2\varepsilon\{k_1(k_1-1)\}^{1/2}, x_1-\varepsilon]$, $h_\varepsilon^{(1)}(x)$ is an arbitrary interpolation between $h_\varepsilon^{(0)}$ and zero providing smooth (up to the second derivatives) conjugation and such that $0 \leqslant h_\varepsilon^{(1)} \leqslant O(\varepsilon^k)$, $|(h_\varepsilon^{(1)})'| \leqslant O(\varepsilon^{k-1})$, $|(h_\varepsilon^{(1)})''| \leqslant O(\varepsilon^{k-2})$ in this domain.

Then we define h_ε as $h_\varepsilon(x) \equiv h_\varepsilon^{(1)}(x)/\int h_\varepsilon^{(1)}\,dx$ (because for h_ε to originate a Markov operator it is necessary that $\int h_\varepsilon\,dx = 1$).

Now let us explain the second condition, which may seem strange. It means $\operatorname{supp} h_\varepsilon \subseteq [x_0+\varepsilon, x_1-\varepsilon]$, and is to ensure that $\operatorname{supp} w_\varepsilon \subseteq [x_0, x_1] \equiv \operatorname{supp} w$, because the action of the convolution operator (4.1) expands the support by ε in either side. And, since $\operatorname{supp} w_\varepsilon \subseteq \operatorname{supp} w$, we have

$$\operatorname{supp} W_\varepsilon \rho \subseteq \operatorname{supp} W\rho \tag{4.7}$$

for any $\rho \geqslant 0$, which will be important later.

One can easily see that for ε small enough, h_ε defined above is nonnegative and normalized: $\int h_\varepsilon\,dx = 1$. It is also smooth and bounded. Indeed, by construction, $h_\varepsilon^{(1)}$ is close to $h_\varepsilon^{(0)}(x)$:

$$\left|\frac{d^n}{dx^n}(h_\varepsilon^{(0)} - h_\varepsilon^{(1)})\right| \leqslant O(\varepsilon^{k-n}) \leqslant O(\varepsilon^{4-n}), \qquad n = 0, 1, 2$$

so $\int h_\varepsilon^{(1)}\,dx = \int h_\varepsilon^{(0)}\,dx + O(\varepsilon^4) = 1 + O(\varepsilon^4)$, and $h_\varepsilon(x) = [1 + O(\varepsilon^4)]\,h_\varepsilon^{(1)}(x)$, where the term $O(\varepsilon^4)$ is *independent* of x. Therefore $h_\varepsilon^{(0)}$ and h_ε are also close:

$$\left|\frac{d^n}{dx^n}(h_\varepsilon^{(0)} - h_\varepsilon)\right| \leqslant O(\varepsilon^{k-n}) \leqslant O(\varepsilon^{4-n}), \qquad n = 0, 1, 2$$

so estimating $h_\varepsilon^{(0)} \equiv w - \varepsilon^2 w''/6$ and its derivatives by means of (4.2), we conclude that for ε small enough

$$0 \leqslant h_\varepsilon \leqslant \mathscr{W}, \qquad |h_\varepsilon'| \leqslant \mathscr{W}, \qquad |h_\varepsilon''| \leqslant \mathscr{W} \tag{4.8}$$

where $\mathscr{W}$ is the constant from (4.2). Then,

$$|w_\varepsilon^{(0)}(x) - w_\varepsilon(x)| = \left| \int [h_\varepsilon^{(0)}(y) - h_\varepsilon(y)]\, q_\varepsilon(x-y)\, dy \right|$$
$$\leqslant O(\varepsilon^4) \int q_\varepsilon(\xi)\, d\xi = O(\varepsilon^4)$$

which together with (4.3) gives $|w_\varepsilon - w| \leqslant O(\varepsilon^3)$, so for ε small enough

$$|w_\varepsilon - w| \leqslant C_1 \varepsilon^3 \tag{4.9}$$

Here and below ε is assumed to be small enough for (4.8)–(4.9) to be satisfied.

The results also hold when supp w is not the single interval, but a union of a finite number of them: suffice it to expand w in a sum $w = \sum_j w_j$ with supp $w_j = [x_0^{(j)}, x_1^{(j)}]$. Then for each w_j we construct corresponding $h_{\varepsilon,j}$ as described above; due to the linearity, $h_\varepsilon = \sum_j h_{\varepsilon,j}$. Since their supports do not overlap and each term satisfies (4.8), the sum h_ε also satisfies it.

Finally, since w is smooth, its support is at any rate a union of intervals, though, perhaps, of an *infinite* number of them. In this case we split supp w into the union of a *finite* number of intervals and the remainder $\mathscr{R}$ so that the Lebesgue measure of the latter (i.e., the total length of remaining intervals) is small: $m(\mathscr{R}) \leqslant O(\varepsilon)$. By construction, the boundaries of $\mathscr{R}$ are also the boundaries of supp w, and as w is smooth, this means that on them $w = w' = \cdots = w^{(\text{iv})} = 0$. Using the exact Taylor expansion with the Lagrange remainder and estimating $w^{(\text{iv})}$ in the intermediate point by (4.2), we see that $w \leqslant \mathscr{W}_4 (m(\mathscr{R}))^4/24 \leqslant O(\varepsilon^4)$ on $\mathscr{R}$. Now replace w with $\bar{w}$, coinciding with $w(x)$ everywhere but on $\mathscr{R}$, where $\bar{w} \equiv 0$. Obviously supp $\bar{w}$ is a union of a finite number of intervals and $|\bar{w} - w| \leqslant O(\varepsilon^4)$, which enables us to construct h_ε, as described above, for $\bar{w}$ instead of w. We omit the details.

5. CONTRACTION PROPERTIES OF THE OPERATOR W

In this section we elaborate the idea of (3.4)–(3.5), which together with the proximity of W to the product operator $W_\varepsilon = Q_\varepsilon H_\varepsilon$, enables us to prove that

$$\int \frac{((W \Delta\rho)(x))^2}{(W\rho)(x)}\, dx \leqslant \kappa^2 \int \frac{[\Delta\rho(y)]^2}{\rho(y)}\, dy, \qquad \kappa = \text{const} < 1$$

for any distribution $\rho(x)$ with a bounded support and deviation $\Delta\rho(x)$ such that $\int \Delta\rho\, dx = 0$ for any interval from those composing $\operatorname{supp} \rho = \bigcup_{i=1}^{N} I_i$ and $\int \{[\Delta\rho(x)]^2/\rho(x)\}\, dx$ exists.

Let us first assume that this interval is unique (and with no zeros of $\rho(x)$ inside, otherwise it should be treated as two adjacent intervals). Assume also that supp w is a unique interval too. Later, in the end of this section, we will eliminate these limitations.

For convenience we denote $W\rho \equiv \hat{\rho}$, $W\,\Delta\rho \equiv \Delta\hat{\rho}$, and, correspondingly, $W_\varepsilon \rho \equiv \hat{\rho}_\varepsilon$, $W_\varepsilon\, \Delta\rho \equiv \Delta\hat{\rho}_\varepsilon$. Let also $\breve{\rho}_\varepsilon \equiv H_\varepsilon \rho$, $\Delta\breve{\rho}_\varepsilon \equiv H_\varepsilon\, \Delta\rho$, so that $\rho_\varepsilon = Q_\varepsilon \breve{\rho}_\varepsilon$, $\Delta\hat{\rho}_\varepsilon = Q_\varepsilon\, \Delta\breve{\rho}_\varepsilon$. We begin with the obvious identity

$$
\begin{aligned}
\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)} = &\left[\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)} - \frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\right] \\
&+ \left[\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)} - \int \frac{[\Delta\rho(y)]^2}{\rho(y)}\, w_\varepsilon(x-y)\, dy\right] \\
&+ \int \frac{[\Delta\rho(y)]^2}{\rho(y)}\, [w_\varepsilon(x-y) - w(x-y)]\, dy \\
&+ \int \frac{[\Delta\rho(y)]^2}{\rho(y)}\, w(x-y)\, dy
\end{aligned} \tag{5.1}
$$

whose second term can be evaluated so that to "extract" the action of Q_ε. Indeed, by the general estimate (2.3),

$$
\frac{(\Delta\breve{\rho}_\varepsilon(y))^2}{\breve{\rho}_\varepsilon(y)} \equiv \frac{((H_\varepsilon\, \Delta\rho)(y))^2}{(H_\varepsilon \rho)(y)} \leqslant \int \frac{[\Delta\rho(z)]^2}{\rho(z)}\, h_\varepsilon(y-z)\, dz
$$

so,

$$
\begin{aligned}
\int \frac{(\Delta\breve{\rho}_\varepsilon(y))^2}{\breve{\rho}_\varepsilon(y)}\, q_\varepsilon(x-y)\, dy &\leqslant \int \frac{[\Delta\rho(z)]^2}{\rho(z)}\, h_\varepsilon(y-z)\, q_\varepsilon(x-y)\, dy\, dz \\
&= \int \frac{[\Delta\rho(z)]^2}{\rho(z)}\, w_\varepsilon(x-z)\, dz
\end{aligned}
$$

Substituting it in (5.1) and estimating $w_\varepsilon - w$ by means of (4.9), we get

$$
\begin{aligned}
\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)} \leqslant &\left[\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)} - \frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\right] \\
&+ \left[\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)} - \int \frac{(\Delta\breve{\rho}_\varepsilon(y))^2}{\breve{\rho}_\varepsilon(y))}\, q_\varepsilon(x-y)\, dy\right] \\
&+ C_1 \varepsilon^3 \int \frac{[\Delta\rho(y)]^2}{\rho(y)}\, dy + \int \frac{[\Delta\rho(y)]^2}{\rho(y)}\, w(x-y)\, dy
\end{aligned} \tag{5.2}
$$

where the second term is exactly what was estimated in (3.5).

Now the idea is as follows. We split the domain $D \equiv \operatorname{supp} \hat{\rho}$ in two parts: $D_d^>$, where $\hat{\rho}(x) \geqslant d$, and the remainder $D_d^< \equiv \overline{D \backslash D_d^>}$. It follows from (4.9) that

$$|\hat{\rho}(x) - \hat{\rho}_\varepsilon(x)| = \left| \int [w_\varepsilon(x-y) - w(x-y)]\, \rho(y)\, dy \right| \leqslant C_1 \varepsilon^3 \quad (5.3)$$

Hence $\hat{\rho}_\varepsilon(x) \geqslant \hat{\rho}(x) - C_1 \varepsilon^3$. Let here and below $\varepsilon \leqslant (d/2C_1)^{1/3}$, then $\hat{\rho}_\varepsilon(x) \geqslant \hat{\rho}(x) - d/2$ and so inside $D_d^>$, $\hat{\rho}_\varepsilon$ is also bounded from below:

$$\hat{\rho}_\varepsilon(x) \geqslant \hat{\rho}(x)/2 \geqslant d/2, \qquad x \in D_d^> \quad (5.4)$$

Thus we can use (5.2) in $D_d^>$, where the denominators in the first term are bounded from below and we expect that for ε small enough this term is $O(\varepsilon^3)$. Outside $D_d^>$ the fractions of (5.2) may diverge, so instead of (5.2) we use the general estimate (2.3):

$$\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)} \leqslant \int \frac{[\Delta\rho(y)]^2}{\rho(y)}\, w(x-y)\, dy$$

which together with (5.2) leads after integration over D to

$$\begin{aligned} \int_D \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\, dx \leqslant {} & [1 + m(D)\, C_1 \varepsilon^3] \int \frac{[\Delta\rho(y)]^2}{\rho(y)}\, dy \\ & + \int_{D_d^>} \left| \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)} - \frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)} \right| dx \\ & + \int_{D_d^>} \left[\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)} - \int \frac{(\Delta\check{\rho}_\varepsilon(y))^2}{\check{\rho}_\varepsilon(y)}\, q_\varepsilon(x-y)\, dy \right] dx \end{aligned} \quad (5.5)$$

where $m(D) \equiv \int_D dx$ is the Lebesgue measure. Since by (2.5) $\operatorname{supp} [\Delta\hat{\rho}]^2/\hat{\rho} \subseteq \operatorname{supp} \hat{\rho} \equiv D$, one can replace the integral $\int_D (\cdot)\, dx$ in the L.H.S. with $\int (\cdot)\, dx$:

$$\begin{aligned} \int \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\, dx \leqslant {} & [1 + m(D)\, C_1 \varepsilon^3] \int \frac{[\Delta\rho(y)]^2}{\rho(y)}\, dy \\ & + \int_{D_d^>} \left| \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)} - \frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)} \right| dx \\ & + \int_{D_d^>} \left[\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)} - \int \frac{(\Delta\check{\rho}_\varepsilon(y))^2}{\check{\rho}_\varepsilon(y)}\, q_\varepsilon(x-y)\, dy \right] dx \end{aligned} \quad (5.6)$$

Due to (4.9) the second term is expected to be $O(\varepsilon^3)$, while (3.7) suggests that the third one is $-O(\varepsilon^2)$. So for ε small enough it dominates and

$$\int \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\,dx \leqslant \int \frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy - O(\varepsilon^2)$$

which is what we need.

5.1. Estimation of the Second Term of (5.6)

To do this we use the identity

$$\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)} - \frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)} \equiv \frac{\Delta\hat{\rho}(x) - \Delta\hat{\rho}_\varepsilon(x)}{(\hat{\rho}_\varepsilon(x))^{1/2}} \cdot \frac{\Delta\hat{\rho}(x) + \Delta\hat{\rho}_\varepsilon(x)}{(\hat{\rho}_\varepsilon(x))^{1/2}} + \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)} \cdot \frac{\hat{\rho}_\varepsilon(x) - \hat{\rho}(x)}{\hat{\rho}_\varepsilon(x)} \tag{5.7}$$

The difference $\hat{\rho}(x) - \hat{\rho}_\varepsilon(x)$ entering it was estimated in (5.3). Similarly,

$$|\Delta\hat{\rho}(x) - \Delta\hat{\rho}_\varepsilon(x)| \leqslant C_1\varepsilon^3 \int |\Delta\rho(y)|\,dy$$

and since by the Cauchy–Bunjakowsky inequality

$$\left[\int |\Delta\rho(y)|\,dy\right]^2 \leqslant \left[\int \rho(y)\,dy\right] \int \frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy$$

we have

$$|\Delta\hat{\rho}(x) - \Delta\hat{\rho}_\varepsilon(x)| \leqslant C_1\varepsilon^3 \left(\int \frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy\right)^{1/2} \tag{5.8}$$

because ρ is a distribution and so $\int \rho\,dx = 1$. As the kernels w, w_ε, and h_ε satisfy $\int w\,dx = 1$, $\int w_\varepsilon\,dx = 1$, $\int h_\varepsilon\,dx = 1$, it readily follows that $\int \hat{\rho}\,dx = 1$, $\int \hat{\rho}_\varepsilon\,dx = 1$, $\int \tilde{\rho}_\varepsilon\,dx = 1$, which we will use below without special remarks. Finally, the simple inequality $(a+b)^2 \leqslant 2(a^2+b^2)$ results in

$$|\Delta\hat{\rho}(x) + \Delta\hat{\rho}_\varepsilon(x)| \leqslant \sqrt{2} \cdot [(\Delta\hat{\rho}(x))^2 + (\Delta\hat{\rho}_\varepsilon(x))^2]^{1/2}$$

so inside $D_d^>$, where according to (5.4), $\hat{\rho}_\varepsilon(x) \geqslant \hat{\rho}(x)/2$, we have

$$\left|\frac{\Delta\hat{\rho}(x) + \Delta\hat{\rho}_\varepsilon(x)}{(\hat{\rho}_\varepsilon(x))^{1/2}}\right| \leqslant \sqrt{2} \cdot \left(2\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)} + \frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\right)^{1/2}, \qquad x \in D_d^>$$

Substituting this inequality together with (5.3) and (5.8) in (5.7) and using the estimate (5.4), we find that for $x \in D_d^>$

$$\left|\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}-\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\right|$$

$$\leqslant 2C_1\varepsilon^3\left\{\left(\frac{1}{d}\int\frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy\right)^{1/2}\left(2\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}+\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\right)^{1/2}+\frac{1}{d}\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\right\}$$

thus

$$\int_{D_d^>}\left|\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}-\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\right|dx$$

$$\leqslant 2C_1\varepsilon^3\left\{\left(\frac{1}{d}\int\frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy\right)^{1/2}\int_D\left(2\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}+\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\right)^{1/2}d\chi+\frac{1}{d}\int\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\,dx\right\} \tag{5.9}$$

Its first term can be estimated by the Cauchy–Bunjakowsky inequality as

$$\int_D\left(2\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}+\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\right)^{1/2}dx$$

$$\leqslant\left[m(D)\int_D\left(2\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}+\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\right)dx\right]^{1/2}$$

$$\leqslant[m(D)]^{1/2}\left[2\int\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\,dx+\int\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\,dx\right]^{1/2}$$

[as usual $m(\cdot)$ is the Lebesgue measure]. Then, according to (2.4),

$$\int\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\,dx\leqslant\int\frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy,\qquad\int\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\,dx\leqslant\int\frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy$$

so finally (5.9) becomes

$$\int_{D_d^>}\left|\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}-\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\right|dx$$

$$\leqslant 2C_1\varepsilon^3\left[\frac{1}{d}+\left(\frac{3m(D)}{d}\right)^{1/2}\right]\int\frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy \tag{5.10}$$

Later we will need that d be small, so that $D_d^>$ to comprise almost all D. So it is possible to assume (just for the sake of convenience) that $d \leqslant 1$. In this case $\sqrt{d} \leqslant d$ and (5.10) can be rewritten as

$$\int_{D_d^>} \left| \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)} - \frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)} \right| dx \leqslant \frac{C_2 \varepsilon^3}{d} \int \frac{[\Delta\rho(y)]^2}{\rho(y)} dy \qquad (5.11)$$

where $C_2 \equiv 2C_1\{1 + [3m(D)]^{1/2}\}$. Substituting this estimate in (5.6), we arrive at

$$\int \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)} dx \leqslant \left(1 + \frac{C_3 \varepsilon^3}{d}\right) \int \frac{[\Delta\rho(y)]^2}{\rho(y)} dy$$
$$+ \int_{D_d^>} \left[\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)} - \int \frac{(\Delta\check{\rho}_\varepsilon(y))^2}{\check{\rho}_\varepsilon(y)} q_\varepsilon(x-y)\, dy \right] dx \qquad (5.12)$$

where $C_3 = C_2 + m(D)\, C_1$.

As $\hat{\rho}_\varepsilon = Q_\varepsilon \check{\rho}_\varepsilon$, $\Delta\hat{\rho}_\varepsilon = Q_\varepsilon \Delta\check{\rho}_\varepsilon$, the last term in this expression is associated with the action of the operator Q_ε and will be estimated in the next subsection using the idea of (3.5).

5.2. Estimation of the Last Item in (5.12) and Contraction Properties of the Operator Q_ε

To implement the idea of (3.5), we have to ascertain that $u \equiv \Delta\check{\rho}_\varepsilon / \check{\rho}_\varepsilon$ is smooth enough to admit the Taylor expansion.

By definition,

$$\check{\rho}_\varepsilon(x) = \int h_\varepsilon(x-y)\, \rho(y)\, dy$$

$$\Delta\check{\rho}_\varepsilon(x) = \int h_\varepsilon(x-y)\, \Delta\rho(y)\, dy$$

For ε small enough (which we assume is satisfied) h_ε has continuous derivatives up to the second, satisfying (4.8):

$$\left| \frac{d^n}{dx^n} h_\varepsilon(x) \right| \leqslant \mathscr{W}, \qquad n = 0, 1, 2$$

from which it follows that $\check{\rho}_\varepsilon(x)$ and $\Delta\check{\rho}_\varepsilon(x)$ also have continuous derivatives up to the second. Indeed,

$$\left| \frac{d^n}{dx^n} \check{\rho}_\varepsilon(x) \right| = \left| \int h_\varepsilon^{(n)}(x-y)\, \rho(y)\, dy \right| \leqslant \mathscr{W} \int \rho(y)\, dy = \mathscr{W}, \qquad n = 0, 1, 2 \qquad (5.13)$$

and similarly $|(d^n/dx^n)\,\Delta\check{\rho}_\varepsilon(x)| \leqslant \mathscr{W} \int |\Delta\rho(y)|\,dy$, which by the Cauchy–Bunjakowsky inequality becomes [cf. (5.8)]

$$\left|\frac{d^n}{dx^n}\,\Delta\check{\rho}_\varepsilon(x)\right| \leqslant \mathscr{W}\cdot\left(\int \frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy\right)^{1/2}, \qquad n=0,1,2 \tag{5.14}$$

Finally, let us show that $\check{\rho}_\varepsilon(x)$ does not vanish in the ε-neighborhood of $D_d^>$, which together with (5.13)–(5.14) implies that in this domain $u \equiv \Delta\check{\rho}_\varepsilon/\check{\rho}_\varepsilon$ is smooth enough. Indeed, by definition

$$\hat{\rho}_\varepsilon(x) \equiv \int q_\varepsilon(x-y)\,\check{\rho}_\varepsilon(y)\,dy = \int q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi$$

Expanding $\check{\rho}_\varepsilon(x+\xi)$ in the *exact* Taylor series with the Lagrange remainder

$$\check{\rho}_\varepsilon(x+\xi) = \check{\rho}_\varepsilon(x) + \xi\check{\rho}'_\varepsilon(x) + \xi^2\check{\rho}''_\varepsilon(x+\xi^*[x,\xi])/2$$

we get [$\int \xi q_\varepsilon(\xi)\,d\xi = 0$ owing to the symmetry of q_ε]

$$\hat{\rho}_\varepsilon(x) = \check{\rho}_\varepsilon(x) + \tfrac{1}{2}\int \xi^2\check{\rho}''_\varepsilon(x+\xi^*[x,\xi])\,d\xi$$

and estimating $\check{\rho}''_\varepsilon$ by means of (5.13), we arrive at $|\hat{\rho}_\varepsilon(x) - \check{\rho}_\varepsilon(x)| \leqslant \varepsilon^2\mathscr{W}/6$, which together with (5.3) gives

$$|\check{\rho}_\varepsilon(x) - \hat{\rho}(x)| \leqslant \varepsilon^2\mathscr{W}/6 + \varepsilon^3 C_1 \tag{5.15}$$

Obviously, in the ε-neighborhood of $D_d^>$

$$\check{\rho}_\varepsilon(x) \geqslant \inf_{D_d^>} \check{\rho}_\varepsilon - \varepsilon\cdot\sup|\check{\rho}'_\varepsilon|$$

so using (5.13) and (5.15), one obtains

$$\check{\rho}_\varepsilon(x) \geqslant d - \frac{\varepsilon^2\mathscr{W}}{6} - \varepsilon^3 C_1 - \varepsilon\mathscr{W} = d - \varepsilon\mathscr{W} - \frac{1}{6\mathscr{W}}(\varepsilon\mathscr{W})^2 - \frac{C_1}{\mathscr{W}^3}(\varepsilon\mathscr{W})^3 \tag{5.16}$$

Let here and below

$$\varepsilon \leqslant \frac{1}{4}\,\frac{d}{\mathscr{W}}\,\frac{1}{1+1/6\mathscr{W}+C_1/\mathscr{W}^3} \tag{5.17}$$

Then obviously $\varepsilon\mathscr{W} < 1$ (it was assumed above that $d \leqslant 1$) and one easily calculates that (5.15)–(5.16) imply

$$|\check{\rho}_\varepsilon(x) - \hat{\rho}(x)| \leqslant d/2 \tag{5.18}$$

$$\check{\rho}_\varepsilon(x) \geqslant d/2, \qquad x \in \varepsilon\text{-neighborhood of } D_d^> \tag{5.19}$$

Finally we notice that from (5.17) it follows that $\varepsilon^3 C_1 \leqslant d/2$, and so (5.17) *includes* the limitation $\varepsilon \leqslant (d/2C_1)^{1/3}$ used in the derivation of (5.4).

Now the estimates (5.13), (5.14), and (5.19) imply that inside the ε-neighborhood of $D_d^>$, $u(x)$ has continuous derivatives up to the second, and it is easy to calculate that

$$|u'(x)| \leqslant \left[\frac{\mathscr{W}}{d/2} + \left(\frac{\mathscr{W}}{d/2}\right)^2\right]\left[\int \frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy\right]^{1/2}$$

$$|u''(x)| \leqslant \left[\frac{\mathscr{W}}{d/2} + 3\left(\frac{\mathscr{W}}{d/2}\right)^2 + 2\left(\frac{\mathscr{W}}{d/2}\right)^3\right]\left[\int \frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy\right]^{1/2}$$

Let here and below $d \leqslant 2\mathscr{W}$, which is just for the sake of convenience, since now the above inequalities can be written as

$$\left.\begin{aligned} |u'(x)| &\leqslant 2\left(\frac{\mathscr{W}}{d/2}\right)^2 \left(\int \frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy\right)^{1/2} \\ |u''(x)| &\leqslant 6\left(\frac{\mathscr{W}}{d/2}\right)^3 \left(\int \frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy\right)^{1/2} \end{aligned}\right\}, \qquad x \in \varepsilon\text{-neighborhood of } D_d^> \tag{5.20}$$

Now we can begin the estimation of the last term in (5.12), which is

$$\int_{D_d^>} \left[\frac{(\Delta\hat\rho_\varepsilon(x))^2}{\hat\rho_\varepsilon(x)} - \int \frac{(\Delta\check\rho_\varepsilon(y))^2}{\check\rho_\varepsilon(y)}\, q_\varepsilon(x-y)\,dy\right] dx \tag{5.21}$$

By definition,

$$\hat\rho_\varepsilon(x) \equiv (Q_\varepsilon \check\rho_\varepsilon)(x) = \int q_\varepsilon(\xi)\,\check\rho_\varepsilon(x+\xi)\,d\xi$$

$$\begin{aligned}\Delta\hat\rho_\varepsilon(x) \equiv (Q_\varepsilon\,\Delta\check\rho_\varepsilon)(x) &= \int q_\varepsilon(\xi)\,\Delta\check\rho_\varepsilon(x+\xi)\,d\xi \\ &= \int u(x+\xi)\,q_\varepsilon(\xi)\,\check\rho_\varepsilon(x+\xi)\,d\xi\end{aligned}$$

and for $x \in D_d^>$, $x+\xi$ lies in the ε-neighborhood of $D_d^>$, so $u(x+\xi)$ admits expansion in the exact Taylor series

$$u(x+\xi) = u(x) + \xi u'(x) + \xi^2 u''(x + \xi^*[x,\xi])/2 \tag{5.22}$$

using which we get

$$\begin{aligned}\Delta\hat\rho_\varepsilon(x) = u(x)\,\hat\rho_\varepsilon(x) &+ u'(x)\int \xi q_\varepsilon(\xi)\,\check\rho_\varepsilon(x+\xi)\,d\xi \\ &+ \tfrac{1}{2}\int \xi^2 q_\varepsilon(\xi)\,u''(x+\xi^*[x,\xi])\,\check\rho_\varepsilon(x+\xi)\,d\xi\end{aligned}$$

Squaring and dividing by $\hat{\rho}_\varepsilon(x)$, we obtain the first term from (5.21):

$$\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)} = [u(x)]^2\,\hat{\rho}_\varepsilon(x) + \frac{[u'(x)]^2}{\hat{\rho}_\varepsilon(x)}\left[\int \xi q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi\right]^2$$
$$+\frac{1}{4\hat{\rho}_\varepsilon(x)}\left[\int \xi^2 q_\varepsilon(\xi)\,u''(x+\xi^*)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi\right]^2$$
$$+2u(x)\,u'(x)\int \xi q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi$$
$$+u(x)\int \xi^2 q_\varepsilon(\xi)\,u''(x+\xi^*)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi$$
$$+\frac{u'(x)}{\hat{\rho}_\varepsilon(x)}\left[\int \xi q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi\right]$$
$$\times\int \xi^2 q_\varepsilon(\xi)\,u''(x+\xi^*)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi \tag{5.23}$$

The second term of the integrand from (5.21) is

$$\int \frac{(\Delta\check{\rho}_\varepsilon(y))^2}{\check{\rho}_\varepsilon(y)}\,q_\varepsilon(x-y)\,dy = \int (u(x+\xi))^2\,q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi$$

and substituting for $u(x+\xi)$ the expansion (5.22), one easily calculates that

$$\int \frac{(\Delta\check{\rho}_\varepsilon(y))^2}{\check{\rho}_\varepsilon(y)}\,q_\varepsilon(x-y)\,dy = [u(x)]^2\,\hat{\rho}_\varepsilon(x) + (u'(x))^2\int \xi^2 q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi$$
$$+\frac{1}{4}\int [\xi^2 u''(x+\xi^*)]^2\,q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi$$
$$+2u(x)\,u'(x)\int \xi q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi$$
$$+u(x)\int \xi^2 q_\varepsilon(\xi)\,u''(x+\xi^*)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi$$
$$+u'(x)\int \xi^3 q_\varepsilon(\xi)\,u''(x+\xi^*)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi \tag{5.24}$$

By the Cauchy–Bunjakowsky inequality,

$$\left[\int \xi^2 u''(x+\xi^*)\,q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi\right]^2$$
$$\leqslant \hat{\rho}_\varepsilon(x)\int [\xi^2 u''(x+\xi^*)]^2\,q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi$$

so comparing (5.23) with (5.24) and estimating the third term in (5.23) by the above inequality, we arrive at

$$\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)} - \int \frac{(\Delta\check{\rho}_\varepsilon(y))^2}{\check{\rho}_\varepsilon(y)} q_\varepsilon(x-y)\,dy$$

$$\leqslant -(u'(x))^2 \int \xi^2 q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi$$

$$+\frac{[u'(x)]^2}{\hat{\rho}_\varepsilon(x)}\left[\int \xi q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi\right]^2$$

$$+\frac{u'(x)}{\hat{\rho}_\varepsilon(x)}\left[\int \xi q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi\right]\int \xi^2 q_\varepsilon(\xi)\,u''(x+\xi^*)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi$$

$$-u'(x)\int \xi^3 q_\varepsilon(\xi)\,u''(x+\xi^*)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi$$

As already mentioned, when $x \in D_d^>$, both $x+\xi$ and $x+\xi^* \in [x, x+\xi]$ belong to the ε-neighborhood of $D_d^>$, and so we can estimate u' and u'' by means of (5.20), which gives

$$\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)} - \int \frac{(\Delta\check{\rho}_\varepsilon(y))^2}{\check{\rho}_\varepsilon(y)} q_\varepsilon(x-y)\,dy$$

$$\leqslant -(u'(x))^2\left\{\int \xi^2 q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi - \frac{1}{\hat{\rho}_\varepsilon(x)}\left[\int \xi q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi\right]^2\right\}$$

$$+\frac{384\mathscr{W}^5}{d^5}\left\{\frac{1}{\hat{\rho}_\varepsilon(x)}\left[\int |\xi|\, q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi\right]\int \xi^2 q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi\right.$$

$$\left.+\int |\xi|^3\, q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi\right\}\int \frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy, \qquad x \in D_d^>$$

Obviously,

$$\int |\xi|^n\, q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi \leqslant \varepsilon^n \int q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi = \varepsilon^n \hat{\rho}_\varepsilon(x)$$

Thus

$$\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)} - \int \frac{(\Delta\check{\rho}_\varepsilon(y))^2}{\check{\rho}_\varepsilon(y)} q_\varepsilon(x-y)\,dy$$

$$\leqslant -(u'(x))^2\left[\int \xi^2 q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi - \varepsilon\left|\int \xi q_\varepsilon(\xi)\,\check{\rho}_\varepsilon(x+\xi)\,d\xi\right|\right]$$

$$+\frac{768\mathscr{W}^5}{d^5}\varepsilon^3\hat{\rho}_\varepsilon(x)\int \frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy, \qquad x \in D_d^> \qquad (5.25)$$

Expanding $\breve{\rho}_\varepsilon(x+\xi)$ in the exact Taylor series

$$\breve{\rho}_\varepsilon(x+\xi)=\breve{\rho}_\varepsilon(x)+\xi\breve{\rho}'_\varepsilon(x+\xi^{**}[x,\xi])$$

and estimating $\breve{\rho}'_\varepsilon$ by means of (5.13), one easily calculates that

$$\left|\int \xi q_\varepsilon(\xi)\,\breve{\rho}_\varepsilon(x+\xi)\,d\xi\right| = \left|\int \xi^2 q_\varepsilon(\xi)\,\breve{\rho}'_\varepsilon(x+\xi^{**})\,d\xi\right|$$

$$\leqslant \mathscr{W}\int \xi^2 q_\varepsilon(\xi)\,d\xi=\varepsilon^2\mathscr{W}/3$$

Then, when $x\in D_d^>$, $x+\xi$ lies within its ε-neighborhood and (5.19) gives $\breve{\rho}_\varepsilon(x+\xi)\geqslant d/2$, so

$$\int \xi^2 q_\varepsilon(\xi)\,\breve{\rho}_\varepsilon(x+\xi)\,d\xi\geqslant\frac{d}{6}\varepsilon^2,\qquad x\in D_d^>$$

and (5.25) becomes

$$\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}-\int\frac{(\Delta\breve{\rho}_\varepsilon(y))^2}{\breve{\rho}_\varepsilon(y)}q_\varepsilon(x-y)\,dy$$

$$\leqslant -(u'(x))^2\frac{\varepsilon^2}{6}(d-2\varepsilon\mathscr{W})+\frac{768\mathscr{W}^5}{d^5}\varepsilon^3\hat{\rho}_\varepsilon(x)\int\frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy,\qquad x\in D_d^>$$

Since our limitation on ε in (5.17) implies $\varepsilon\leqslant d/4\mathscr{W}$, this becomes

$$\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}-\int\frac{(\Delta\breve{\rho}_\varepsilon(y))^2}{\breve{\rho}_\varepsilon(y)}q_\varepsilon(x-y)\,dy$$

$$\leqslant -\varepsilon^2\frac{d}{12}(u'(x))^2+\frac{768\mathscr{W}^5}{d^5}\varepsilon^3\hat{\rho}_\varepsilon(x)\int\frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy,\qquad x\in D_d^>$$

Integrating over $D_d^>$ and taking into account that

$$\int_{D_d^>}\hat{\rho}_\varepsilon(x)\,dx\leqslant\int\hat{\rho}_\varepsilon(x)\,dx=1$$

we arrive at

$$\int_{D_d^>}\left[\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}-\int\frac{(\Delta\breve{\rho}_\varepsilon(y))^2}{\breve{\rho}_\varepsilon(y)}q_\varepsilon(x-y)\,dy\right]dx$$

$$\leqslant -\frac{\varepsilon^2 d}{12}\int_{D_d^>}(u'(x))^2\,dx+\frac{768\mathscr{W}^5}{d^5}\varepsilon^3\int\frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy \tag{5.26}$$

Now we have to estimate $\int_{D_d^>} (u')^2\, dx$. Notice that since the kernel w is smooth and the supports of w and ρ are intervals, the domain $D \equiv \operatorname{supp} \hat{\rho} \equiv \operatorname{supp} W\rho$ is also an interval with no zeros of $\hat{\rho}$ inside. Therefore the domain $D_d^>$, where $\hat{\rho}(x) \geqslant d$, is interval too. Let us introduce

$$v(x) \equiv u(x) - \frac{\int_{D_d^>} u(x)\, \check{\rho}_\varepsilon(x)\, dx}{\int_{D_d^>} \check{\rho}_\varepsilon(x)\, dx}$$

By construction

$$\int_{D_d^>} v(x)\, \check{\rho}_\varepsilon(x)\, dx = 0$$

so $v(x)$ cannot but changes sign in the interval $D_d^>$, and since it is continuous inside this domain, there exists some $a \in D_d^>$ such that $v(a) = 0$. This enables us to write

$$v(x) = \int_a^x v'(y)\, dy$$

and applying the Cauchy–Bunjakowsky inequality, one finds that for $x \in D_d^>$

$$v^2(x) \leqslant \left(\int_a^x dy \right) \left[\int_a^x (v'(y))^2\, dy \right] \leqslant m(D) \int_{D_d^>} (v'(y))^2\, dy$$

where, as usual, $m(\cdot)$ is the Lebesgue measure. Since $\int_{D_d^>} \check{\rho}_\varepsilon\, dx \leqslant \int \check{\rho}_\varepsilon\, dx = 1$, we immediately obtain

$$\int_{D_d^>} v^2(x)\, \check{\rho}_\varepsilon(x)\, dx \leqslant m(D) \int_{D_d^>} (v'(y))^2\, dy$$

or, substituting for u and v their definitions,

$$\int_{D_d^>} (u'(x))^2\, dx \geqslant \frac{1}{m(D)} \left(\int_{D_d^>} \frac{(\Delta \check{\rho}_\varepsilon(x))^2}{\check{\rho}_\varepsilon(x)}\, dx - \frac{\left(\int_{D_d^>} \Delta \check{\rho}_\varepsilon(x)\, dx\right)^2}{\int_{D_d^>} \check{\rho}_\varepsilon(x)\, dx} \right) \tag{5.27}$$

By (2.5), $\operatorname{supp}\{[\Delta \check{\rho}_\varepsilon]^2/\check{\rho}_\varepsilon\} \subseteq \operatorname{supp} \check{\rho}_\varepsilon$ and obviously $\operatorname{supp} \Delta \check{\rho}_\varepsilon \subseteq \operatorname{supp} \check{\rho}_\varepsilon$. Then, since the action of the convolution operator Q_ε *expands* the support, $\operatorname{supp} \check{\rho}_\varepsilon \subset \operatorname{supp} Q_\varepsilon \check{\rho}_\varepsilon \equiv \operatorname{supp} \hat{\rho}_\varepsilon$. Finally, according to (4.7), $\operatorname{supp} \hat{\rho}_\varepsilon \subseteq \operatorname{supp} \hat{\rho} \equiv D$, and so

$$\operatorname{supp} \frac{[\Delta \check{\rho}_\varepsilon]^2}{\check{\rho}_\varepsilon} \subset D, \qquad \operatorname{supp} \Delta \check{\rho}_\varepsilon \subset D, \qquad \operatorname{supp} \check{\rho}_\varepsilon \subset D \tag{5.28}$$

Thus

$$1=\int \breve{\rho}_\varepsilon\,dx=\int_{D_d^>}\breve{\rho}_\varepsilon\,dx+\int_{D_d^<}\breve{\rho}_\varepsilon\,dx$$

or

$$\int_{D_d^>}\breve{\rho}_\varepsilon\,dx=1-\int_{D_d^<}\breve{\rho}_\varepsilon\,dx$$

By definition of $D_d^<$, inside it, $0\leqslant\hat{\rho}\leqslant d$, and since, according to (5.18), $|\breve{\rho}_\varepsilon-\hat{\rho}|\leqslant d/2$, we have $0\leqslant\breve{\rho}_\varepsilon\leqslant\frac{3}{2}d$ ($\breve{\rho}_\varepsilon$ is not negative). Therefore,

$$\int_{D_d^>}\breve{\rho}_\varepsilon\,dx=1-\int_{D_d^<}\breve{\rho}_\varepsilon\,dx\geqslant 1-\tfrac{3}{2}d\cdot m(D_d^<)\geqslant 1-\tfrac{3}{2}d\cdot m(D)$$

so assuming that here and below $d\leqslant 1/3m(D)$, we arrive at

$$\int_{D_d^>}\breve{\rho}_\varepsilon\,dx\geqslant\tfrac{1}{2}\tag{5.29}$$

Now let us estimate the term $\int_{D_d^>}\Delta\breve{\rho}_\varepsilon\,dx$ from (5.27). Since $\int\Delta\rho\,dx=0$, we have

$$\int\Delta\breve{\rho}_\varepsilon(x)\,dx=\int dx\int h_\varepsilon(x-y)\,\Delta\rho(y)\,dy=\int\Delta\rho(y)\,dy=0$$

which due to (5.28) gives $\int_{D_d^>}\Delta\breve{\rho}_\varepsilon\,dx=-\int_{D_d^<}\Delta\breve{\rho}_\varepsilon\,dx$. Estimating the latter integral by the Cauchy–Bunjakowsky inequality, and recalling that $\int\breve{\rho}_\varepsilon\,dx=1$, we get

$$\begin{aligned}\left(\int_{D_d^>}\Delta\breve{\rho}_\varepsilon\,dx\right)^2&=\left(\int_{D_d^<}\Delta\breve{\rho}_\varepsilon\,dx\right)^2\\&\leqslant\left[\int_{D_d^<}\left(\frac{\Delta\breve{\rho}_\varepsilon}{\breve{\rho}_\varepsilon}\right)^2\breve{\rho}_\varepsilon\,dx\right]\int_{D_d^<}\breve{\rho}_\varepsilon\,dx\leqslant\int_{D_d^<}\frac{(\Delta\breve{\rho}_\varepsilon(x))^2}{\breve{\rho}_\varepsilon(x)}\,dx\end{aligned}$$

and substituting this estimate and (5.29) in (5.27), we arrive at

$$\int_{D_d^>}(u'(x))^2\,dx\geqslant\frac{1}{m(D)}\left(\int_{D_d^>}\frac{(\Delta\breve{\rho}_\varepsilon(x))^2}{\breve{\rho}_\varepsilon(x)}\,dx-2\int_{D_d^<}\frac{(\Delta\breve{\rho}_\varepsilon(x))^2}{\breve{\rho}_\varepsilon(x)}\,dx\right)$$

The first embedding from (5.28) implies that

$$\int\frac{(\Delta\breve{\rho}_\varepsilon(x))^2}{\breve{\rho}_\varepsilon(x)}\,dx=\int_D\frac{(\Delta\breve{\rho}_\varepsilon(x))^2}{\breve{\rho}_\varepsilon(x)}\,dx=\int_{D_d^<}\frac{(\Delta\breve{\rho}_\varepsilon(x))^2}{\breve{\rho}_\varepsilon(x)}\,dx+\int_{D_d^>}\frac{(\Delta\breve{\rho}_\varepsilon(x))^2}{\breve{\rho}_\varepsilon(x)}\,dx$$

Thus

$$\int_{D_d^>} (u'(x))^2\, dx \geqslant \frac{1}{m(D)}\left(\int \frac{(\Delta\check{\rho}_\varepsilon(x))^2}{\check{\rho}_\varepsilon(x)}\, dx - 3\int_{D_d^<} \frac{(\Delta\check{\rho}_\varepsilon(x))^2}{\check{\rho}_\varepsilon(x)}\, dx\right) \quad (5.30)$$

Due to the general estimate (2.4)

$$\int \frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\, dx \equiv \int \frac{(Q_\varepsilon \circ \Delta\check{\rho}_\varepsilon(x))^2}{Q_\varepsilon \circ \check{\rho}_\varepsilon(x)}\, dx \leqslant \int \frac{(\Delta\check{\rho}_\varepsilon(x))^2}{\check{\rho}_\varepsilon(x)}\, dx$$

so (5.30) can be transformed to

$$\int_{D_d^>} (u'(x))^2\, dx \geqslant \frac{1}{m(D)}\left(\int_{D_d^>} \frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\, dx - 3\int_{D_d^<} \frac{(\Delta\check{\rho}_\varepsilon(x))^2}{\check{\rho}_\varepsilon(x)}\, dx\right) \quad (5.31)$$

Using the identity

$$\begin{aligned}\int_{D_d^>} \frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\, dx &\equiv \int_{D_d^>} \left(\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)} - \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\right) dx + \int_{D_d^>} \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\, dx \\ &= \int_{D_d^>} \left(\frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)} - \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\right) dx \\ &\quad + \int \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\, dx - \int_{D_d^<} \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\, dx\end{aligned}$$

and estimating its first term by means of (5.11), we get

$$\int_{D_d^>} \frac{(\Delta\hat{\rho}_\varepsilon(x))^2}{\hat{\rho}_\varepsilon(x)}\, dx \geqslant -\frac{C_2\varepsilon^3}{d}\int \frac{(\Delta\rho(x))^2}{\rho(x)}\, dx + \int \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\, dx - \int_{D_d^<} \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\, dx$$

so (5.31) becomes

$$\begin{aligned}\int_{D_d^>} (u'(x))^2\, dx \geqslant \frac{1}{m(D)}\Bigg(&\int \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\, dx - \int_{D_d^<} \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)}\, dx \\ &- 3\int_{D_d^<} \frac{(\Delta\check{\rho}_\varepsilon(x))^2}{\check{\rho}_\varepsilon(x)}\, dx - \frac{C_2\varepsilon^3}{d}\int \frac{(\Delta\rho(x))^2}{\rho(x)}\, dx\Bigg) \quad (5.32)\end{aligned}$$

Applying the general estimate (2.3) to $\check{\rho}_\varepsilon \equiv H_\varepsilon \rho$, $\Delta\check{\rho}_\varepsilon \equiv H_\varepsilon\, \Delta\rho$, one obtains

$$\frac{(\Delta\check{\rho}_\varepsilon(x))^2}{\check{\rho}_\varepsilon(x)} \leqslant \int \frac{(\Delta\rho(y))^2}{\rho(y)} h_\varepsilon(x-y)\, dy$$

and similarly for $\hat{\rho} \equiv W\rho$, $\Delta\hat{\rho} \equiv W\Delta\rho$,

$$\frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)} \leqslant \int \frac{(\Delta\rho(y))^2}{\rho(y)} w(x-y)\, dy$$

The kernels w and h_ε (for ε small enough) are bounded: $0 \leqslant w \leqslant \mathscr{W}$, $0 \leqslant h_\varepsilon \leqslant \mathscr{W}$; see (4.2), (4.8). Therefore,

$$\int_{D_d^<} \frac{(\Delta\check{\rho}_\varepsilon(x))^2}{\check{\rho}_\varepsilon(x)} dx \leqslant \int_{D_d^<} dx \cdot \mathscr{W} \int \frac{(\Delta\rho(y))^2}{\rho(y)} dy$$
$$= m(D_d^<) \cdot \mathscr{W} \int \frac{(\Delta\rho(y))^2}{\rho(y)} dy$$
$$\int_{D_d^<} \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)} dx \leqslant m(D_d^<) \cdot \mathscr{W} \int \frac{(\Delta\rho(y))^2}{\rho(y)} dy$$

and (5.32) becomes

$$\int_{D_d^>} (u'(x))^2\, dx$$
$$\geqslant \frac{1}{m(D)} \left\{ \int \frac{(\Delta\hat{\rho}(x))^2}{\hat{\rho}(x)} dx - \left[4\mathscr{W} m(D_d^<) + \frac{C_2\varepsilon^3}{d} \right] \int \frac{(\Delta\rho(x))^2}{\rho(x)} dx \right\}$$

Recall that $D_d^>$ is the subset of D where $0 \leqslant \hat{\rho} \leqslant d$. So for $d \leqslant d'$, $D_d^< \subseteq D_{d'}^<$ and thus the Lebesgue measure $m(D_d^<)$ is a *monotone* nonincreasing function. Moreover, it vanishes as $d \to 0$:

$$m(D_d^<) \xrightarrow{d \to 0} 0$$

Indeed, consider a sequence $d_1 \geqslant d_2 \geqslant \cdots \geqslant 0$ such that $d_n \to 0$. The sequence of closed embedded sets $D \supseteq D_{d_1}^< \supseteq D_{d_2}^< \supseteq \cdots$ has a limit $D_0^< = \bigcap_n D_{d_n}^<$, which is the domain where $0 \leqslant \hat{\rho}(x) \leqslant \min_n d_n$, that is, where $\hat{\rho}(x) = 0$. On the other hand, $D_0^< \subseteq D$, while by definition of

$$D \equiv \operatorname{supp} \hat{\rho} \equiv \overline{\{x \mid \hat{\rho}(x) > 0\}}$$

each its subset where $\hat{\rho}(x) = 0$ has zero Lebesgue measure. Therefore $m(D_0^<) = 0$ and using the continuity of the Lebesgue measure, we conclude that $\lim_{n\to\infty} m(D_{d_n}^<) = m(D_0^<) = 0$.

This enables to take d_0 so small (it should also satisfy the previous limitations $d_0 \leqslant 1$ and $d_0 \leqslant 2\mathscr{W}$) that $m(D_{d_0}^<) \leqslant 1/8\mathscr{W}$, which gives

$$\int_{D_{d_0}^{>}} (u'(x))^2\,dx$$

$$\geqslant \frac{1}{m(D)}\left[\int \frac{(\Delta\hat\rho(x))^2}{\hat\rho(x)}\,dx - \left(\frac{1}{2}+\frac{C_2\varepsilon^3}{d_0}\right)\int \frac{(\Delta\rho(x))^2}{\rho(x)}\,dx\right]$$

and substituting this estimate in (5.26), we arrive at

$$\int_{D_{d_0}^{>}} \left[\frac{(\Delta\hat\rho_\varepsilon(x))^2}{\hat\rho_\varepsilon(x)} - \int \frac{(\Delta\check\rho_\varepsilon(y))^2}{\check\rho_\varepsilon(y)}\,q_\varepsilon(x-y)\,dy\right]dx$$

$$\leqslant -\frac{\varepsilon^2 d_0}{12m(D)}\int \frac{(\Delta\hat\rho(x))^2}{\hat\rho(x)}\,dx$$

$$+\left[\frac{\varepsilon^2 d_0}{12m(D)}\left(\frac{1}{2}+\frac{C_2\varepsilon^3}{d_0}\right)+\frac{768\mathscr{W}^5}{d_0^5}\varepsilon^3\right]\int \frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy$$

and (5.12) becomes

$$\int \frac{(\Delta\hat\rho(x))^2}{\hat\rho(x)}\,dx$$

$$\leqslant -\frac{\varepsilon^2 d_0}{12m(D)}\int \frac{(\Delta\hat\rho(x))^2}{\hat\rho(x)}\,dx$$

$$+\left(1+\frac{\varepsilon^2 d_0}{24m(D)}+C_3\frac{\varepsilon^3}{d_0}+\frac{768\mathscr{W}^5}{d_0^5}\varepsilon^3+\frac{C_2\varepsilon^5}{12m(D)}\right)\int \frac{[\Delta\rho(y)]^2}{\rho(y)}\,dy$$

or, denoting

$$C_4\equiv\frac{d_0}{12m(D)},\qquad C_5\equiv\frac{768\mathscr{W}^5}{d_0^5}+\frac{C_3}{d_0},\qquad C_6\equiv\frac{C_2}{12m(D)}$$

and recalling that $\hat\rho\equiv W\rho$, $\Delta\hat\rho\equiv W\,\Delta\rho$:

$$\int \frac{((W\,\Delta\rho)(x))^2}{(W\rho)(x)}\,dx \leqslant \frac{1+\varepsilon^2 C_4/2+\varepsilon^3 C_5+\varepsilon^5 C_6}{1+\varepsilon^2 C_4}\int \frac{[\Delta\rho(x)]^2}{\rho(x)}\,dx$$

$$=\left(1-\varepsilon^2\frac{C_4/2-\varepsilon C_5-\varepsilon^3 C_6}{1+\varepsilon^2 C_4}\right)\int \frac{[\Delta\rho(x)]^2}{\rho(x)}\,dx$$

which for ε small enough results in

$$\int \frac{((W\,\Delta\rho)(x))^2}{(W\rho)(x)}\,dx \leqslant \left(1-\frac{\varepsilon^2 C_4}{4}\right)\int \frac{[\Delta\rho(x)]^2}{\rho(x)}\,dx$$

Since ε and d are *auxiliary* parameters of which these integrals are independent, we conclude that the above inequality means the following:

If w has a bounded support and satisfies (4.2) and supp $W\rho$ is bounded, then there exists $\kappa = \kappa[\rho] < 1$ such that

$$\int \frac{((W\Delta\rho)(x))^2}{(W\rho)(x)}\,dx \leqslant \kappa^2 \int \frac{[\Delta\rho(x)]^2}{\rho(x)}\,dx \tag{5.33}$$

for any $\Delta\rho$ for which the r.h.s. exists and $\int \Delta\rho\,dx = 0$. If these conditions are not satisfied, then κ may reach 1, but never exceeds it; see (2.4).

Now let us briefly consider the case when supp ρ and/or supp w consist of *several* intervals. Those composing supp ρ will be denoted as I_i. Expand w and ρ in the sums of ρ_i and w_j whose supports are intervals: $\rho(x) = \sum_i \rho_i(x)$, $w(x) = \sum_j w_j(x)$; obviously $\rho_i(x) = \chi_{I_i}(x)\,\rho(x)$, where $\chi_A(x)$ is the indicator of the set A, i.e., 1 for $x \in A$ and 0 otherwise. The existence of the integral $\int ([\Delta\rho]^2/\rho)\,dx$ implies that $\Delta\rho \equiv 0$ outside supp $\rho = \bigcup_i I_i$, thus $\Delta\rho$ admits expansion $\Delta\rho(x) = \sum_i \chi_{I_i}(x)\,\Delta\rho(x) \equiv \sum_i \Delta\rho_i(x)$. Due to the linearity

$$\hat{\rho}(x) \equiv (W\rho)(x) = \sum_{i,j} (W_j\rho_i)(x) \equiv \sum_{i,j} \hat{\rho}_{ij}(x)$$

$$\Delta\hat{\rho}(x) \equiv (W\Delta\rho)(x) = \sum_{i,j} (W_j\,\Delta\rho_i)(x) \equiv \sum_{i,j} \Delta\hat{\rho}_{ij}(x)$$

where W_j is the convolution operator with the kernel w_j.

Let $\int_{I_i} \Delta\rho\,dx \equiv \int \Delta\rho_i\,dx = 0$. Since by definition both supp w_j and supp ρ_i are unique intervals, we can estimate $W_j\,\Delta\rho_i$ by means of (5.33):

$$\int \frac{[(W_j\,\Delta\rho_i)(x)]^2}{(W_j\rho_i)(x)}\,dx \leqslant \left(\int w_j\,dx\right)\cdot \kappa^2[\rho_i, w_j]\cdot \int \frac{[\Delta\rho_i(x)]^2}{\rho_i(x)}\,dx \tag{5.34}$$

[It is obvious why the factor $\int w_j\,dx$ arises: (5.33) requires that the integral of the kernel be 1, which is not satisfied for w_j. Using the normalized kernel $w_j/\int w_j\,dx$ we immediately arrive at the above estimate].

Now, by the Cauchy–Bunjakowsky inequality,

$$(\Delta\hat{\rho})^2 = \left(\sum_{i,j} \Delta\hat{\rho}_{ij}\right)^2 = \left(\sum_{i,j} \frac{\Delta\hat{\rho}_{ij}}{\hat{\rho}_{ij}}\cdot \hat{\rho}_{ij}\right)^2$$

$$\leqslant \left(\sum_{i,j} \hat{\rho}_{ij}\right)\cdot \sum_{i,j}\left(\frac{\Delta\hat{\rho}_{ij}}{\hat{\rho}_{ij}}\right)^2 \cdot \hat{\rho}_{ij} = \hat{\rho}\cdot \sum_{i,j} \frac{(\Delta\hat{\rho}_{ij})^2}{\hat{\rho}_{ij}}$$

so

$$\int \frac{[\Delta\hat{\rho}(x)]^2}{\hat{\rho}(x)}\,dx \leqslant \sum_{i,j} \frac{[\Delta\hat{\rho}_{ij}(x)]^2}{\hat{\rho}_{ij}(x)}\,dx$$

which using (5.34) becomes

$$\int \frac{[(W\Delta\rho)(x)]^2}{(W\rho)(x)}\,dx \leqslant \sum_{i,j}\left(\left(\int w_j\,dx\right)\cdot \kappa^2[\rho_i, w_j]\cdot \int \frac{[\Delta\rho_i(x)]^2}{\rho_i(x)}\,dx\right)$$

where each $\kappa[\rho_i, w_j]$ is <1. If the number of intervals composing the supports of ρ and w is finite, then obviously $\kappa[\rho]\equiv \max_{ij}\kappa[\rho_i, w_j]<1$, and recalling that $\sum_j w_j = w$ which integral is 1, we obtain

$$\begin{aligned}\int \frac{[(W\Delta\rho)(x)]^2}{(W\rho)(x)}\,dx &\leqslant \kappa^2[\rho]\cdot\sum_i \int \frac{[\Delta\rho_i(x)]^2}{\rho_i(x)}\,dx \\ &\equiv \kappa^2[\rho]\cdot\sum_i \int \chi_{I_i}(x)\cdot\frac{[\Delta\rho(x)]^2}{\rho(x)}\,dx \\ &= \kappa^2[\rho]\cdot\int \frac{[\Delta\rho(x)]^2}{\rho(x)}\,dx\end{aligned}$$

so the estimate (5.33) holds in this case as well.

6. APPLICATION TO THE NOISY MAPS

Let us return to the noisy map (1.1), which is assumed to have a bounded attractor $\mathscr{A}$. This implies that if $\operatorname{supp} p\subseteq\mathscr{A}$, then $\operatorname{supp}\mathscr{L}_{f,w}p\subseteq\mathscr{A}$, etc. Now let us take a deviation of a distribution, i.e., $\Delta p(x)$ with $\int \Delta p\,dx = 0$ and such that $\int\{[\Delta p]^2/p\}\,dx$ exists. Denoting $\rho\equiv\mathscr{L}_f p$, $\Delta\rho\equiv\mathscr{L}_f\Delta p$, we obtain from (2.6)

$$\int \frac{[\Delta\rho(x)]^2}{\rho(x)}\,dx \leqslant \int \frac{[\Delta p(x)]^2}{p(x)}\,dx \tag{6.1}$$

If $\operatorname{supp} p\subseteq\mathscr{A}$, then $W\rho$ has a finite support, because $\mathscr{L}_{f,w} = W\mathscr{L}_f$ and so $W\rho = \mathscr{L}_{f,w}p$. Then $\mathscr{L}_f$, as a Markov operator, conserves the total measure, thus $\int\Delta\rho\,dx = \int\Delta p\,dx = 0$. Therefore the conditions of Section 5 are satisfied and by (5.33)

$$\int \frac{((W\Delta\rho)(x))^2}{(W\rho)(x)}\,dx \leqslant \kappa^2 \int \frac{[\Delta\rho(x)]^2}{\rho(x)}\,dx$$

Combining this with (6.1) and substituting for ρ and $\Delta\rho$ their definitions, we get

$$\int \frac{((\mathscr{L}_{f,w}\Delta p)(x))^2}{(\mathscr{L}_{f,w}p)(x)}\,dx \leqslant \kappa^2 \int \frac{[\Delta p(x)]^2}{p(x)}\,dx \tag{6.2}$$

where $\kappa = \kappa[p] < 1$ if supp p consists of a finite number of intervals I_i, on each of them $\int_{I_i} \Delta p\, dx = 0$ [then this property holds for ρ too and so (5.33) is satisfied]. If we take for p the invariant distribution $P = \mathscr{L}_{f,w} P$ (obviously supp $P = \mathscr{A}$) and use the norm (1.5), this becomes

$$\|\mathscr{L}_{f,w}\, \Delta p\| \leqslant \kappa\, \|\Delta p\| \tag{6.3}$$

with $\kappa \equiv \kappa[P] < 1$ (if the above conditions are *not* satisfied, then κ may reach 1, but never exceeds it). The role of the map f therefore consists in providing the invariant distribution P with *bounded* support.

A more mathematical formulation is as follows:

Let RL_2 be the linear space endowed with the norm (1.5): $\|g\|^2 \equiv \int (g^2/P)\, dx$. Owing to (6.3), $\mathscr{L}_{f,w}$ is a bounded linear operator $RL_2 \mapsto RL_2$ (while usually it is considered as $L_1 \mapsto L_1$), $\|\mathscr{L}_{f,w}\, g\| \leqslant \|g\|$. Denote by $RL_2^{(0)}$ its subspace consisting of those $\Delta p \in RL_2$ for which $\int_{I_i} \Delta p\, dx$ for any I_i from those composing supp $P = 0$. It is invariant under the action of $\mathscr{L}_{f,w}$, and if the distribution of noise $w(\cdot)$ is smooth enough [i.e., satisfies (4.2)] and its support is bounded, then in $RL_2^{(0)}$, $\mathscr{L}_{f,w}$ is contracting: $\|\mathscr{L}_{f,w}\, \Delta p\| \leqslant \kappa\, \|\Delta p\|$ for some $\kappa < 1$.

This means that if we denote by $p_n(x)$ the distribution in the map (1.1) at the nth iteration, then iterates of two different distributions p_0 and $\tilde{p}_0$ which integrals over any I_i from these composing supp P coincide exponentially converge:

$$\|\tilde{p}_n - p_n\| \leqslant \kappa^n\, \|\tilde{p}_0 - p_0\| \tag{6.4}$$

7. ON L_2-TYPE NORMS FOR SPATIOTEMPORAL CHAOS

In the previous part of this paper the norm (1.5) and related integral functionals were used to prove strong contraction properties of Markov operators. It is marvelous that almost the same integrals and norms arise when we study coupled map lattices (CML) and their random fields. This will be briefly discussed in this concluding section.

Consider a CML with a finite coupling range R

$$x_i(n+1) = F(x_{i-R}(n),..., x_{i+R}(n)) \tag{7.1}$$

where i is a lattice point and n a discrete time. Denote the corresponding probability measure (at time n) as $\mu_n(\mathbf{x})$ and its finite-dimensional densities as $P_n^{(L)}(x_i, x_{i+1},..., x_{i+L})$. Owing to the uniformness of the model, its random field $\{x_i\}$ is (statistically) uniform and thus these distributions are independent of i. Let also $p_n^{(L)}(x_i | x_{i+1},..., x_{i+L})$ be the (right) conditional distribution:

$$P_n^{(L)}(x_i,..., x_{i+L}) = p_n^{(L)}(x_i | x_{i+1},..., x_{i+L})\, P_n^{(L-1)}(x_{i+1},..., x_{i+L}) \tag{7.2}$$

Assume that spatial correlations decay in the sense that these conditional distributions are almost independent of far variables in their tails and so are close to the infinite tail one $p_n(x_i \mid x_{i+1},...)$:

$$p_n^{(m)}(x_i \mid x_{i+1},..., x_{i+m}) \xrightarrow{m \to \infty} p_n(x_i \mid x_{i+1},..., x_{i+m},...) \tag{7.3}$$

Now let us derive a useful asymptotic relation between magnitudes of infinitesimal deviations of absolute and conditional distributions. From (7.2) it immediately follows that

$$\begin{aligned} \Delta P_n^{(L)}(x_i,..., x_{i+L}) &= \Delta p_n^{(L)}(x_i \mid x_{i+1},..., x_{i+L}) \cdot P_n^{(L-1)}(x_{i+1},..., x_{i+L}) \\ &\quad + p_n^{(L)}(x_i \mid x_{i+1},..., x_{i+L}) \cdot \Delta P_n^{(L-1)}(x_{i+1},..., x_{i+L}) \end{aligned}$$

and using the equalities

$$\int p_n^{(L)}(x_i \mid x_{i+1},...)\, dx_i = 1, \qquad \int \Delta p_n^{(L)}(x_i \mid x_{i+1},...)\, dx_i = 0$$

one calculates that

$$\begin{aligned} &\int \frac{[\Delta P_n^{(L)}(x_i,..., x_{i+L})]^2}{P_n^{(L)}(x_i,..., x_{i+L})}\, dx_i \cdots dx_{i+L} \\ &\quad = \int \frac{[\Delta P_n^{(L-1)}(x_{i+1},..., x_{i+L})]^2}{P_n^{(L-1)}(x_{i+1},..., x_{i+L})}\, dx_{i+1} \cdots dx_{i+L} \\ &\quad + \int \left(\frac{\Delta p_n^{(L)}(x_i \mid x_{i+1},..., x_{i+L})}{p_n^{(L)}(x_i \mid x_{i+1},..., x_{i+L})} \right)^2 P_n^{(L)}(x_i,..., x_{i+L})\, dx_i \cdots dx_{i+L} \end{aligned} \tag{7.4}$$

Notice that for any function u depending on a *finite* number of variables $u = u(x_i,..., x_{i+m})$

$$\int u\, d\mu_n = \int u(x_i,..., x_{i+m})\, P_n^{(m)}(x_i,..., x_{i+m})\, dx_i \cdots dx_{i+m}$$

so (7.4) can be rewritten as

$$\int \left(\frac{\Delta P_n^{(L)}}{P_n^{(L)}} \right)^2 d\mu_n = \int \left(\frac{\Delta P_n^{(L-1)}}{P_n^{(L-1)}} \right)^2 d\mu_n + \int \left(\frac{\Delta p_n^{(L)}}{p_n^{(L)}} \right)^2 d\mu_n$$

which via iteration leads to

$$\int \left(\frac{\Delta P_n^{(L)}}{P_n^{(L)}} \right)^2 d\mu_n = \sum_{m=0}^{L} \int \left(\frac{\Delta p_n^{(m)}}{p_n^{(m)}} \right)^2 d\mu_n \tag{7.5}$$

(because $p_n^{(0)}$ and $P_n^{(0)}$ are the same function).

The decay of correlations (7.3) implies that

$$\int \left(\frac{\Delta p_n^{(m)}}{p_n^{(m)}}\right)^2 d\mu_n \xrightarrow{m \to \infty} \int \left(\frac{\Delta p_n}{p_n}\right)^2 d\mu_n$$

so dividing both sides of (7.5) by L and taking the limit $L \to \infty$, we get

$$\lim_{L \to \infty} \frac{1}{L} \int \left(\frac{\Delta P_n^{(L)}}{P_n^{(L)}}\right)^2 d\mu_n = \int \left(\frac{\Delta p_n}{p_n}\right)^2 d\mu_n \tag{7.6}$$

or

$$\lim_{L \to \infty} \frac{1}{L} \int \frac{[\Delta P_n^{(L)}(x_i,..., x_{i+L})]^2}{P_n^{(L)}(x_i,..., x_{i+L})} dx_i \cdots dx_{i+L} = \int \left(\frac{\Delta p_n}{p_n}\right)^2 d\mu_n \tag{7.6'}$$

Now let us return to the dynamical system (7.1). Denote for the sake of convenience $\mathbf{x} \equiv \{x_i,..., x_{i+L}\}$, $\mathbf{x}' \equiv \{x_{i-R},..., x_{i+L+R}\}$. The CML (7.1) obviously originates the mapping $\mathbf{x} = \Phi_L(\mathbf{x}')$, which[12] enables us to derive the relation between finite-dimensional distributions:

$$P_{n+1}^{(L)}(\mathbf{x}) = \int \delta(\mathbf{x} - \Phi_L(\mathbf{x}'))\, P_n^{(L+2R)}(\mathbf{x}')\, d\mathbf{x}' \tag{7.7}$$

resembling the ordinary Frobenius–Perron operator (save for the fact that now $\mathbf{x}$ and $\mathbf{x}'$ have different dimensions). So, similarly to (2.4), we obtain

$$\int \frac{[\Delta P_{n+1}^{(L)}(\mathbf{x})]^2}{P_{n+1}^{(L)}(\mathbf{x})} d\mathbf{x} \leqslant \int \frac{[\Delta P_n^{(L+2R)}(\mathbf{x}')]^2}{P_n^{(L+2R)}(\mathbf{x}')} d\mathbf{x}' \tag{7.8}$$

Dividing both sides by L and taking the limit $L \to \infty$, we arrive at, according to (7.6),

$$\int \left(\frac{\Delta p_{n+1}}{p_{n+1}}\right)^2 d\mu_{n+1} \leqslant \int \left(\frac{\Delta p_n}{p_n}\right)^2 d\mu_n \tag{7.9}$$

which means that the action of *any* dynamics with finite coupling range does not increase the magnitude of the (relative) deviation of conditional distributions.

Another advantage of (7.9)-type functionals is that in these "norms" the deviations of left and right conditional distributions coincide:

$$\int \left(\frac{\Delta p_n^{(L)}}{p_n^{(L)}}\right)^2 d\mu_n = \int \left(\frac{\Delta p_n^{(-L)}}{p_n^{(-L)}}\right)^2 d\mu_n$$

where the left conditional distribution $p_n^{(-L)}(x_i | x_{i-1},..., x_{i-L})$ has its "condition tail" on the left of the pivot site.

Notice that if we take for μ_n and p_n the *invariant* measure, then the integrals in (7.5)–(7.9) become conventional weighted L_2 norms

$$\|\Delta P_n^{(L)}\|^2 \equiv \int \left(\frac{\Delta P_n^{(L)}}{P^{(L)}}\right)^2 d\mu, \qquad |||\Delta p_n^{(L)}|||^2 \equiv \int \left(\frac{\Delta p_n^{(L)}}{p^{(L)}}\right)^2 d\mu \tag{7.10}$$

Altogether, though usually one considers distributions as elements of L_1, it appears sometimes useful to work with them in L_2 endowed with (1.5)-type norm for absolute distributions and (7.10)-type norm for conditional ones.

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